

CAUSAL DYNAMICAL TRIANGULATIONS WITH TOROIDAL TOPOLOGY*

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Causal Dynamical Triangulations (CDT) is a background-independent approach to quantum gravity which provides a lattice regularization. In the case of spherical spatial topology, a universe with geometry of a four-sphere emerges dynamically in the so-called de Sitter phase. Imposing toroidal spatial topology changes this picture significantly and the average spatial volume profile becomes constant. Although no background geometry is put in by hand, the full quantum theory of CDT is able to identify a classical background geometry with superimposed quantum fluctuations. We determine the effective action for spatial volume by measuring the covariance matrix and show how to heal the problem of uniform volume profile.

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1. Introduction

This paper studies the effective action of Causal Dynamical Triangulations (CDT) using the covariance matrix method and deals with the problem of uniform volume profile in the case of toroidal spatial topology. The CDT model is a nonperturbative approach to quantum gravity and provides a lattice regularization of the formal gravitational path integral,

$$\int \mathcal{D}\mathcal{M}[g] e^{iS^{\text{EH}}[g]} \longrightarrow \sum_{\mathcal{T}} e^{-S^{\text{R}}[\mathcal{T}]} . \quad (1)$$

The underlying spacetime manifold \mathcal{M} is discretized via four-dimensional triangulations \mathcal{T} built of four-simplices. The discrete counterpart of the Einstein–Hilbert action $S^{\text{EH}}[g] = -\frac{1}{G} \int dt \int d^D x \sqrt{g} (R - 2\Lambda)$ is called Regge action [1] and is given by a linear combination of bare coupling constants

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$\kappa_0, \kappa_4, \Delta$ and a total number of simplices (N_4), number of vertices (N_0) and number of $\{4, 1\}$ simplices [2]

$$S^R[\mathcal{T}] = -\kappa_0 N_0 + \kappa_4 N_4 + \Delta(N_{14} - 6N_0). \quad (2)$$

On a microscopic level, the bare coupling constants are functions of the gravitational constant G , the cosmological constant Λ and an asymmetry factor between lengths of time-like links and space-like links. Additionally, the CDT model introduces a global proper-time foliation of the spacetime manifold $\mathcal{M} = \Sigma \times S^1$. Due to the foliation, the Wick rotation procedure is well-defined leading to real weights in (1). Consequently, we apply Monte Carlo techniques to approximate expectation values of observables. Once the spatial topology Σ is set, it is not allowed to change in time. For convenience, time-periodic boundary conditions are chosen.

In [3], it was shown that the CDT model exhibits a strong dependence on the spatial topology. A basic observable investigated in CDT is a spatial volume n_t defined as a number of tetrahedra building a slice with discrete time coordinate $t = 1, \dots, T$. Figure 1 (left) depicts average volume profiles for spherical ($\Sigma = S^3$) and toroidal spatial topology ($\Sigma = T^3$). In the former case, in phase C , the average volume profile corresponds to a Euclidean de Sitter space [5], which is a maximally symmetric classical solution to Einstein field equations. In the latter case, the volume profile is uniform.

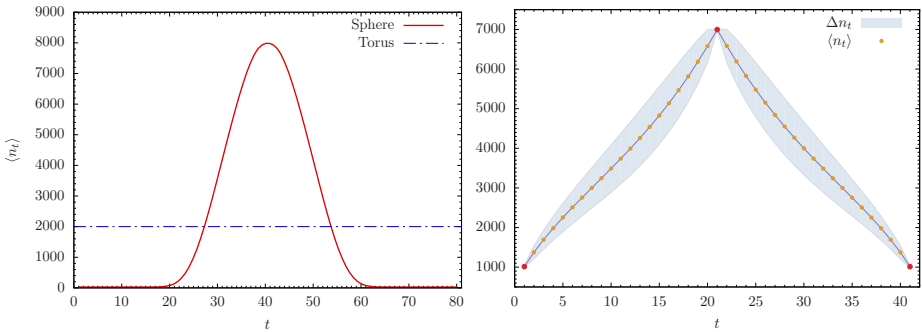


Fig. 1. Left: Average volume profile $\langle n_t \rangle$ for toroidal (dashed line) and spherical (solid line) spatial topology with periodic boundary conditions. Right: Average volume profile for toroidal topology with fixed volumes of slices $t = 1$ and $t = 21$. Total number of slices is $T = 40$.

2. Effective action via the covariance matrix

The effective action $S[n]$ describes behavior of spatial volumes n_t and is obtained by integrating out all degrees of freedom except the scale factor. It is important to note that in the presented approach, no degrees of freedom

are frozen. The covariance matrix method provides information about the effective action and can be used together with Monte Carlo methods. It works well for a spherical topology [9]. However, in the toroidal case with time-periodic boundary conditions, one has to face a problem of the constant in time average volume profile. As a consequence, a single measurement acquires data for a single value of spatial volume. Multiple simulations are necessary to obtain meaningful results introducing additional measurement errors.

To deal with the mentioned problem, we span the volume profile over some range by adding terms to the bare action (2) which partially fix volumes of the first and middle slice, namely $S_{\text{fix}} = \frac{1}{2}\varepsilon[(n_1 - \bar{n}_1)^2 + (n_h - \bar{n}_h)^2]$, $h = T/2 + 1$. Figure 1 (right) outlines the average volume in such scenario with a superimposed amplitude of fluctuations (gray halo). Dots (red) denote time slices with fixed volumes.

The measured distribution of volumes n_t is close to Gaussian and the amplitude of fluctuations is relatively small. Therefore, it seems reasonable to apply a semiclassical approximation, where the effective action is expanded up to quadratic terms in $\eta_t = n_t - \langle n_t \rangle$

$$S[n = \langle n \rangle + \eta] \approx S[\langle n \rangle] + \frac{1}{2}\eta_t P_{tt'} \eta'_t + O(\eta^3), \quad P_{tt'} = \left. \frac{\partial^2 S[n]}{\partial n_t \partial n_{t'}} \right|_{n=\langle n \rangle}. \quad (3)$$

The Sturm–Liouville operator P is directly related to the covariance matrix

$$C_{tt'} = \langle \eta_t \eta_{t'} \rangle = \langle (n_t - \langle n_t \rangle)(n_{t'} - \langle n_{t'} \rangle) \rangle = [P^{-1}]_{tt'}. \quad (4)$$

The covariance matrix C can be measured in Monte Carlo CDT simulations giving information about (second derivative of) the effective action. Taking into account the time-reversal symmetry of the system $n_t \leftrightarrow n_{T-t}$, numerical errors can be further reduced by a symmetrization procedure.

Simulations show that the inverse of matrix C , *i.e.* the matrix P , has a tridiagonal structure, with non-zero elements only on a diagonal and first sub- or superdiagonal. Remaining matrix elements are zero up to numerical noise. Such form suggests that the effective action is quasi-local in time

$$S[n] = \sum_t K(n_t, n_{t+1}) + U(n_t), \quad (5)$$

where the kinetic term $K(x, y)$ couples only adjacent slices and is a single source of non-zero subdiagonal elements. The potential term $U(x)$ contributes only to the diagonal.

2.1. The kinetic term

As a general rule, the kinetic term gives a dominating contribution to the path integral, and thus is extracted with the smallest uncertainty. Since the subdiagonal elements of matrix P depend only on the function K , it has to be determined before the potential term. The kinetic coefficient k_t ,

$$P_{tt+1} = -2k_t = \left. \frac{\partial^2 K(x, y)}{\partial x \partial y} \right|_{x=\langle n_t \rangle, y=\langle n_{t+1} \rangle}, \quad (6)$$

is shown in Fig. 2 (left). The linear dependency $\frac{1}{k_t} = \Gamma (\langle n_t \rangle + \langle n_{t+1} \rangle)$ is clearly visible which implies the following form of the kinetic term

$$K(x, y) = \frac{1}{\Gamma} \frac{(x - y)^2}{x + y}. \quad (7)$$

The same behavior is observed for spherical spatial topology [2].

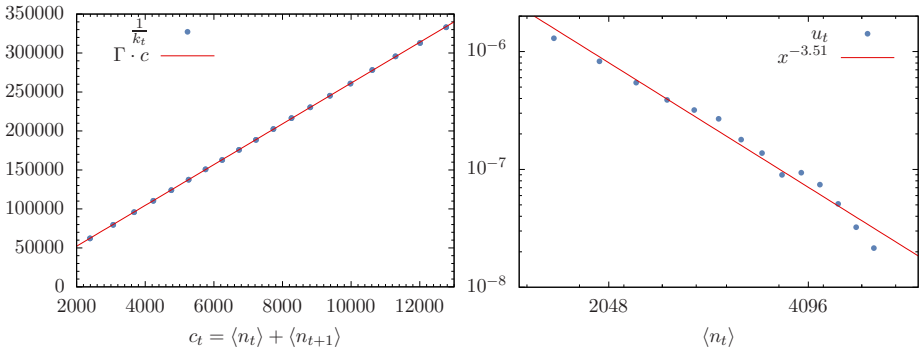


Fig. 2. Left: Inverse of the kinetic coefficient k_t extracted from the covariance matrix as a function of volume $c_t = \langle n_t \rangle + \langle n_{t+1} \rangle$. Right: Log-log plot of the potential coefficient u_t against volume $\langle n_t \rangle$.

2.2. The potential term

The diagonal elements of matrix P depend both on the kinetic and potential term

$$P_{tt} = k_t + k_{t-1} + u_t = \left. \frac{\partial^2 S[n]}{\partial n_t^2} \right|_{n_t=\langle n_t \rangle}, \quad u_t = \left. \frac{\partial^2 U(x)}{\partial x^2} \right|_{x=\langle n_t \rangle}. \quad (8)$$

Knowing k_t (Section 2.1), we can extract the potential coefficients u_t . As can be seen from Fig. 2 (right), the dependence of u_t on $\langle n_t \rangle$ is given by a power-law. It follows from (8) that $U(x) = \mu x^\gamma + \lambda x$, where the measured exponent γ is very close to $-\frac{3}{2}$ and $\mu, \lambda > 0$.

3. Conclusions

The paper presents a method of determining the effective action in a single Monte Carlo simulation, even in the case of toroidal spatial topology. Based on the results, it can be concluded that the continuous version of the effective action is given by

$$S[v] = \int dt \frac{\dot{v}^2}{\Gamma v} + \mu v^{-3/2} + \lambda v,$$

where v is the physical volume. While the kinetic term seems independent of the spatial topology, the difference in the potential is responsible for distinctive average volume profiles. The measured potential $U(v)$ has a single minimum which explains the uniform average volume profile observed for toroidal topology.

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